On the Capacity and Rate of ISI Channels

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***Abstract* — We examine the capacity and rate of ISI channels in additive white Gaussian noise. Previous papers for the 1+*D* or 1–*D* channel showed a minimum E_b/N_0 of –4.6 dB using the water pouring capacity formulas of these channels, 3 dB below the capacity for a flat channel. However, these papers failed to take into account that the channel gain can be greater than one when water pouring is used. We examine this affect in detail, showing that one must be careful how the water pouring equations should be used.**

***Index Terms* — ISI, partial response, capacity, information rate**

I. Introduction

RECENTLY, several papers have presented capacity curves for ISI channels in additive white Gaussian noise (AWGN) that were above capacity for flat channels for low rates [1,2]. In [1], the capacity of the 1+*D* ISI channel (using the water pouring formulas) is shown to be above channel capacity for a flat channel below 1.5 bit/s/Hz. This is despite "normalising" the energy of the channel to 1, by setting the tap amplitudes to $1/\sqrt{2}$. In [2], the minimum E_b/N_0 for the normalised 1–*D* channel was shown to be ln(2)/2 or –4.60 dB, 3 dB below the capacity for a flat channel of ln(2) or –1.6 dB.

We examine the water pouring capacity formulas and the relationship of the transmitter with the channel in both power and bandwidth. If the transmitter is located after an ISI filter for a channel where the power gain is unitary and flat, then it is incorrect to find the capacity of the ISI filter. The capacity for this channel is $C = W\log_2(1+S/N)$ where C is the capacity in bit/s, W is the bandwidth in Hz and S/N is the signal to noise ratio. If the spectrum before the ISI filter is flat, than an information rate below capacity is obtained. If the spectrum before the ISI filter is that for a channel with the same response, then a water pouring rate below capacity is obtained.

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II. Water Pouring

The capacity of an ISI channel with additive noise is given by [3]

$$C(S) = \frac{1}{2\ln(2)} \int_{-\infty}^{\infty} \max\left[0, \ln\frac{\theta}{N(f)/|H(f)|^2}\right] df \tag{1}$$

$$S = \int_{-\infty}^{\infty} \max\left[0, \theta - \frac{N(f)}{|H(f)|^2}\right] df \tag{2}$$

where $C(S)$ is the capacity in bits per second, S is the transmitted power in watts *before* the ISI channel, $N(f)$ is the noise density in watts/Hz added after the ISI channel, and $H(f)$ is the channel spectrum. The process of solving these equations is normally called "*water pouring*". Note that $|H(f)|^2$ represents the power gain with frequency for the channel. This gain can be greater than one for an active channel or less than one for a passive channel. For real channels, one must measure the ratio of power in to power out, which for an electrical channel means measuring both voltage and current.

For example, for the 1+D channel with additive white Gaussian noise we have

$$H(f) = \begin{cases} (1 + e^{-j2\pi fT})/\sqrt{2} & ; -W \le f \le W \\ 0 & ; -W > f > W \end{cases} \tag{3}$$

and $N(f) = N_0/2$ W/Hz. The symbol period is T and with perfect Nyquist signalling $2WT = 1$. For $|f| \le W$ we have $|H(f)|^2 = 1 + \cos(\pi f/W)$ which is plotted in Figure 1 with $W = 1$ Hz. The two and three tap channels have $|H(f)|^2 = (1 + 2\cos(\pi f/W))^2/3$ and $|H(f)|^2 = 2\cos^2(\pi f/W) + 2\cos^3(\pi f/W)$, and are plotted in Figures 2 and 3, respectively. Notice that the peak of frequency response is equal to m, the number of tap settings.

To determine the integration range for a given θ, we let $\theta = N(w)/|H(w)|^2$, where w are the "bowl" edges. For the 1+D channel we have the edges at –w and w where

$$\theta = \frac{N_0/2}{1 + cos(\pi f/w)} \tag{4}$$

and we can simplify (1) and (2) to

$$C(S) = \frac{1}{2\ln(2)} \int_{-w}^{w} \ln(2\theta(1 + \cos(\pi f/W)/N_0)\ df \tag{5}$$

$$S = \int_{-w}^{w} \theta - \frac{N_0/2}{1 + \cos(\pi f/W)}\ df = 2W\theta - \frac{N_0 W}{\pi}\tan\left(\frac{\pi w}{2W}\right). \tag{6}$$

There is no closed form solution for (5), so numerical integration is used. We have that the energy per bit at capacity is $E_b = S/C(S)$. Normalising W to 1 Hz and N_0 to 1 W/Hz, we plot $C(S)$ verses E_b/N_0 in Figure 4. Also plotted is flat channel capacity $C = \log_2(1+CE_b/N_0)$ and the capacities for the three and four tap channels. As can be seen, the $1+D$ channel capacity is above flat channel capacity below 1.5 bit/s/Hz going down to approximately –4.6 dB, 3 dB below the minimum flat channel capacity of –1.6 dB. It can be easily shown that the minimum E_b/N_0 with water pouring is $\ln(2)/|H(f_p)|^2$ where f_p is the frequency at which $|H(f)|^2$ is maximum. For an m tap channel, this is minimised when all the taps are equal giving a minimum E_b/N_0 of $\ln(2)/m$.

In Figure 5 we plot the information rate

$$C(S) = \frac{1}{2}\int_{-W}^{W} \log_2\left(1 + \frac{S|H(f)|^2}{2WN(f)}\right) df \tag{7}$$

for the three channels which assumes a flat transmit power spectrum.

III. Water Pouring Rates

The problem with using the traditional water pouring equations for partial response channels is that the gain of the channel can be greater than one where water pouring is performed. This is due to the bandwidth being restricted to where the channel gain is greatest. For a passive channel, this power gain can not be for free and must be taken into account. To explain what is going on, an electrical schematic of the two tap channel is shown in Figure 6. Two voltage sources in series are used to represent the voltage of the two taps being added together (two current sources added in parallel could also be used). The resistor R represents the load of the channel.

If the delay is such that a frequency is coincident, the power that is transmitted is $2V^2/R$ with each voltage source supplying a power of V^2/R. If the signals are 180° out of phase, zero power is transmitted from the source and zero power is received at the receiver. In general, regardless of the delay, the ratio of transmitted power to the received power is one. That is, the real channel frequency response is $H(f) = 1$. Averaged over the whole phase range, the transmitted (and received) power is V^2/R.

With water pouring, the power S is calculated *before* the ISI channel (which we call the *transmit power*). However, for partial response channels we need to calculate the power *after* the ISI channel, since that is the power that is actually transmitted. This power is

$$S = \int_{-\infty}^{\infty} \max\left[0, \theta|H(f)|^2 - N(f)\right] df. \tag{8}$$

The water pouring rate $C(S)$ ($C(S)$ is no longer a capacity since the actual transmit power is not optimal) is the same as (1), as can be seen by substituting $S|H(f)|^2/2$ with $\max[0, \theta|H(f)|^2 - N(f)]$ in (7). Figure 7 shows the water pouring rates. In this case the rates are below the information rates, indicating that water pouring is not optimum for these channels (as expected, since the true channel response is equal to 1).

A possible application for the capacities in Figure 4 is the multipath channel. For example we could have omnidirectional receive and transmit antennas with a direct and a reflected path. Water pouring will tell us to transmit in those frequencies which have the highest gain. The multipath is allowing us to make more efficient use of the transmitted power and thus is equivalent to an antenna gain. If all the power could be sent from the transmit to the receive antenna, then partial response signalling should not have a greater capacity than flat channel capacity.

The information rate after the ISI channel is given by

$$C(S) = \frac{1}{2} \int_{-W}^{W} \log_2\left(1 + \frac{S|H(f)|^2}{sN(f)}\right) df, \qquad s = \int_{-W}^{W} |H(f)|^2 df. \tag{9}$$

IV. Water Pouring With Respect To Transmit Bandwidth

The water pouring curves assumed that the bandwidth is equal to W. However, if the ISI channel is at the transmitter, then we can normalise the capacity with the actual bandwidth used. For the channels considered, the bandwidth efficiency is $C(S)/w$ (bit/s/Hz) where w is the "bowl" edge. For the three and four tap channels, the side–lobes were cut–off, i.e., the maximum integration limits were set to –2/3 to 2/3 for the three tap channel and –1/2 to 1/2 for the four tap channel. If the side–lobes are included for high SNR, bandwidth efficiency is reduced due to the large increase in bandwidth. Figure 8 plots the water pouring rate with respect to transmit bandwidth. In this case, the rates are very close to each other, but above the information rate and below flat channel capacity.

V. Conclusions

Water pouring for partial response channels must take into account that the channel gain can be greater than one. When the channel gain is taken into account, rates under flat channel capacity are achieved. We also examined rates after the ISI channel and capacity and rates with respect to transmitted bandwidth.

VI. Acknowledgment

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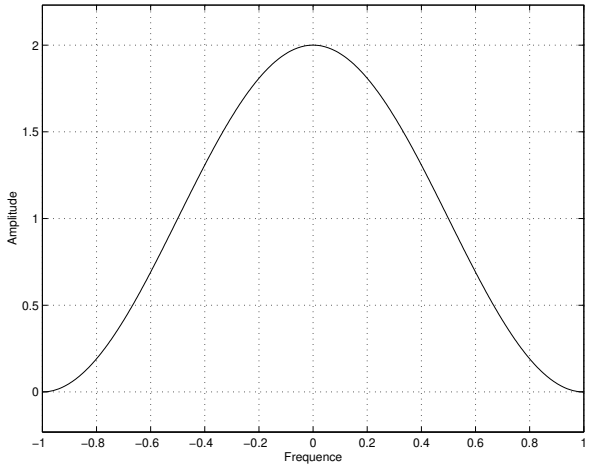

Figure 1: 1+D Channel

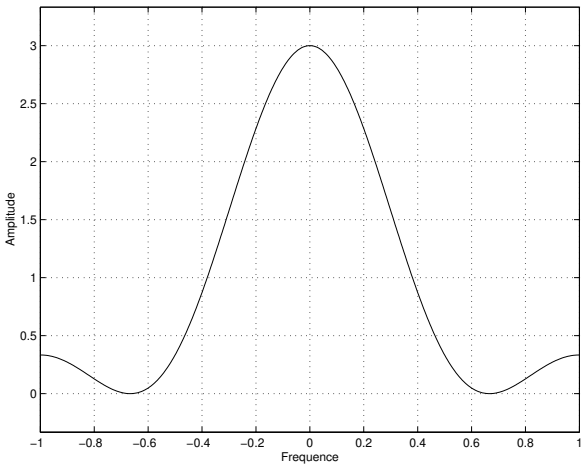

Figure 2: 1+D+D^2 Channel

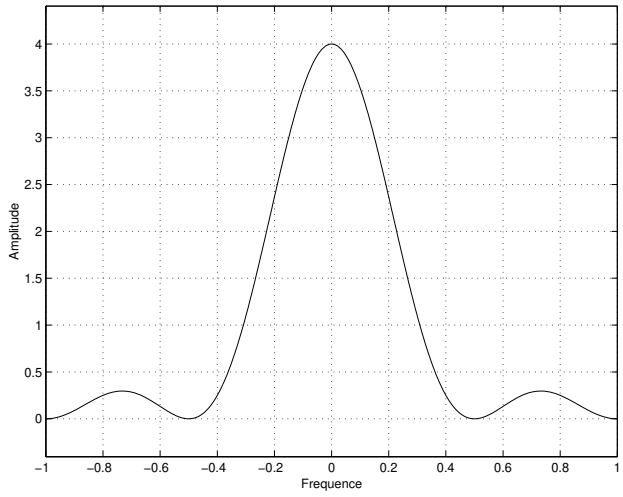

Figure 3: 1+D+D^2+D^3 Channel

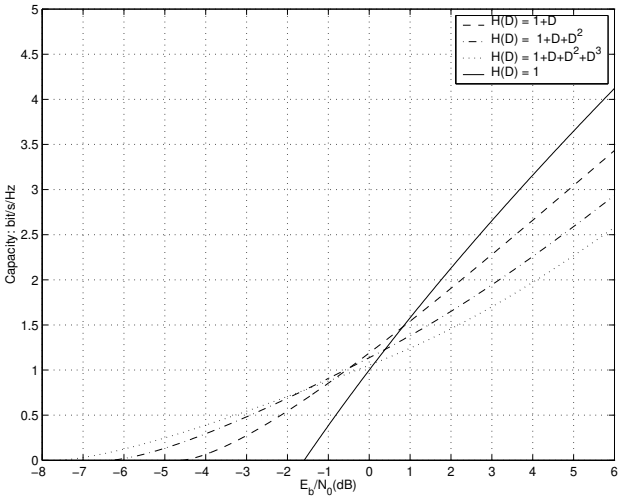

Figure 4: Capacities

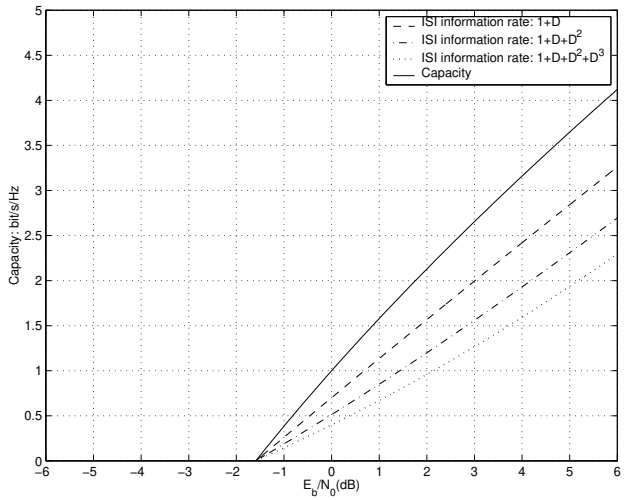

Figure 5: Information rates

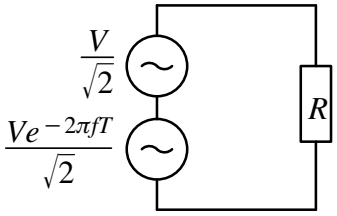

Figure 6: 2–tap channel circuit.

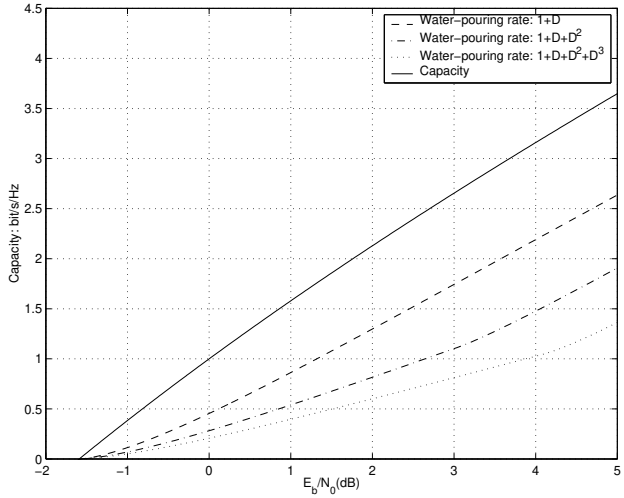

Figure 7: Water pouring rates

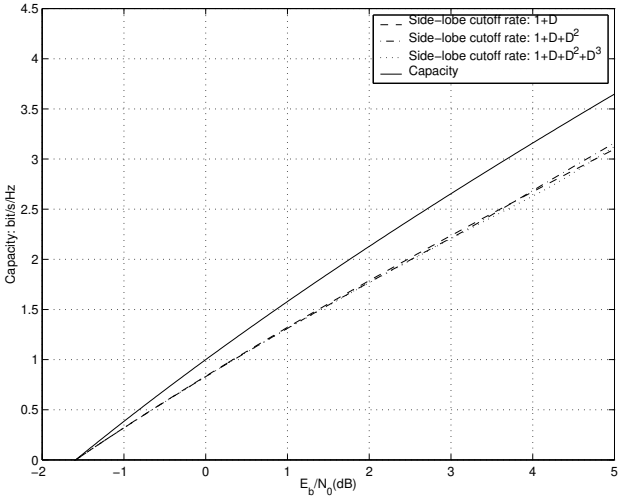

Figure 8: Water pouring rates with respect to transmit bandwidth